JOURNAL OF FUNCTIONAL ANALYSIS 29, 308–318 (1978)

Bounded Approximate Identities, Factorization, and a Convolution Algebra

ALLAN M. SINCLAIR

Department of Mathematics, University of Edinburgh, Edinburgh, EH9 3JZ, Scotland

Communicated by the Editors

Received August 20, 1976

In this paper a Cohen factorization theorem $x = a^t \cdot x_t$ $(t > 0)$ is proved for a Banach algebra A with a bounded approximate identity, where $t \mapsto a^t$ is a continuous one-parameter semigroup in A. This theorem is used to show that a separable Banach algebra B has a bounded approximate identity bounded by 1 if and only if there is a homomorphism θ from $L^1(\mathbb{R}^+)$ into B such that $\| \theta \| = 1$ and $\theta(L^1(\mathbb{R}^+)) . B = B = B \cdot \theta(L^1(\mathbb{R}^+))$. Another corollary is that a separable Banach algebra with bounded approximate identity has a commutative bounded approximate identity, which is bounded by 1 in an equivalent algebra norm.

1. INTRODUCTION

The original form of Cohen's factorization theorem states that if x is an element of a Banach algebra A with a bounded approximate identity, then there is an element a in A such that $x \in aA$ (see [3, 5, 10]). One way this result has been generalized is by replacing $x \in aA$ with $x \in \bigcap_1^\infty a^n A$ [2] (and see [6]). In 1975 B. E. Johnson obtained an unpublished factorization $x = ay$ such that for each positive integer n there is an element $a^{1/n}$ in the algebra with $(a^{1/n})^n = a$. There are related power factorization results in an excellent paper by Esterle [9]. The main theorem in this paper is a result of this type. If x is an element of a Banach algebra A with a bounded approximate identity, then there are functions $t \mapsto a^t$ and $t \mapsto x_t$ from the set of positive real numbers $\mathbb{R}^+$ into A such that $x = a^t x_t$ for all positive t, and $t \mapsto a^t$ is a (norm) continuous one-parameter semigroup (Theorem 1). A stronger form than this is stated and proved, and the growth and limit conditions obtained are important in our applications of the theorem.

Let $L^1(\mathbb{R}^+)$ denote the Banach algebra of complex valued Lebesgue integrable functions on the positive real line $\mathbb{R}^+$ with the convolution product $f * g(t) = \int_0^t f(t - s)\, g(s)\, ds$. We shall assume that all Banach algebras and spaces are over the complex field, and note that with suitable minor changes the results apply to real Banach algebras. Recall that the Banach algebra $L^1(\mathbb{R}^+)$ has a bounded

0022–1236/78/0293–0308$02.00/0

approximate identity. Clearly a Banach algebra B is unital (that is, has an identity of norm 1) if and only if there is a homomorphism θ of norm 1 from the Banach algebra of complex numbers $\mathbb{C}$ into B such that $\theta(\mathbb{C}) \cdot B = B = B \cdot \theta(\mathbb{C})$. We prove that if B is a separable Banach algebra, then B has a bounded approximate identity bounded by 1 if and only if there is a homomorphism θ of norm 1 from the Banach algebra $L^1(\mathbb{R}^+)$ into B such that

$$\theta(L^1(\mathbb{R}^+)) \cdot B = B = B \cdot \theta(L^1(\mathbb{R}^+)).$$

This is a different and deeper characterization of a Banach algebra with a bounded approximate identity than that of [15]. The homomorphism θ we define is the same as that in [13] (see also [11, Theorem 15.2.1]). Several of our conclusions in Theorem 1 could be worded in terms of semigroups [11].

In Theorem 8 we show that a separable Banach algebra with bounded approximate identity has a commutative bounded approximate identity, which is bounded by 1 in an equivalent algebra norm. This is known for C^*-algebras [1] and group algebras [12]. It is also known that a separable commutative Banach algebra with bounded approximate identity has another approximate identity that is bounded by 1 in an equivalent norm [8].

If A is a Banach algebra recall that A has a *bounded approximate identity bounded by d* if for each $\epsilon > 0$ and each finite subset $a_1, \ldots, a_n$ in A there is an e in A with $\| e \| \leqslant d$ and $\| ea_j - a_j \| + \| a_j e - a_j \| < \epsilon$ for $j = 1, \ldots, n$. For a discussion of bounded approximate identities and known results about them, see [10, Section 32] or [3, Section 11], and [2].

2. Main Result

The first six conclusions are the important ones for our applications.

THEOREM 1. *Let A be a Banach algebra with a bounded approximate identity bounded by d, let X be a left Banach A module, and let Y be a right Banach A module. Let H denote the open right half $\{z \in \mathbb{C}\colon \operatorname{Re} z > 0\}$ of the complex plane, let C be a bounded subset of H, let $\delta > 0$, and let $t \mapsto \alpha_t\colon \mathbb{R}^+ \to [1 + \delta, \infty)$ be a continuous function such that $\alpha_t \to \infty$ as $t \to \infty$. If x is in the closed linear span X_0 of $A \cdot X$, and y is in the closed linear span Y_0 of $Y \cdot A$, then there are analytic functions $t \mapsto a^t$, $t \mapsto x_t$, and $t \mapsto y_t$ from H into A, X, and Y such that*

(i) *$x = a^t \cdot x_t$ and $y = y_t \cdot a^t$ for all $t \in H$,*

(ii) *$t \mapsto a^t$ is a homomorphism from the additive semigroup H into the multiplicative semigroup A,*

(iii) *$a^t \cdot x_{s+t} = x_s$ and $y_{s+t} \cdot a^t = y_s$ for all $s, t \in H$,*

(iv) *if $d = 1$, then $\| a^t \| \leqslant 1$ for all $t \in \mathbb{R}^+$,*

(v) $a^t \cdot x \to x$ *and* $y \cdot a^t \to y$ *as* $t \to 0$ $(t \in H)$,

(vi) $\{\| a^t \| : 0 < t \leqslant 1\}$ *is bounded,*

(vii) $\| x - x_t \| \leqslant \delta$ *and* $\| y - y_t \| \leqslant \delta$ *for all* $t \in C$,

(viii) $\| x_t \| \leqslant \alpha_{|t|}{}^{|t|}$ *and* $\| y_t \| \leqslant \alpha_{|t|}{}^{|t|}$ *for all* $t \in H$, *and*

(xi) $x_t \in (A \cdot x)^-$ *and* $y_t \in (y \cdot A)^-$ *for all* $t \in H$.

Remarks 2. The proof is based on Cohen's proof, but with minor variations to help us with powers. The proof is given in the next section and in this section we note where the proof differs from Cohen's proof. Let A_1 be the Banach algebra obtained from A by adjoining an identity in the standard manner. In the usual proof a sequence $b_n = \sum_{j=1}^{n} \lambda(1 - \lambda)^{j-1} e_j + (1 - \lambda)^n$ is defined in A_1 such that b_n is invertible and the sequence $(b_n^{-1} \cdot x)$ is Cauchy in X. The sequence (b_n) converges to an element in A by the choice of λ in relation to d. In the definition of the sequence (b_n) a crucial role is played by the factors of the form $(1 + \lambda(e - 1))$. The difficulty when considering powers in the noncommutative case is that $(uv)^2 \neq u^2v^2$. This difficulty is overcome by using the exponential function, and heuristically replacing the factor $(1 + \lambda(e - 1))$ by $\exp(e - 1)$. We use $b_n = \exp(\sum_1^n (e_j - 1))$. It is easy to obtain a^t from $b_n{}^t$, where we define b^t by $b^t = \exp tf$ for t in $\mathbb{C}$ when b has been defined by $b = \exp f$. In this proof the b_n are clearly invertible and are chosen to be convergent whereas in Cohen's proof the b_n are clearly convergent and are chosen to be invertible. In existing proofs of factorization theorems, and this one, the elements b_n are in the principal component of the group of invertibles of A_1. This is further motivation for the use of the exponential. The elements b_n could be replaced by

$$b_n = \exp\left(\sum_1^n \lambda_j(e_j - 1)\right),$$

where (λ_j) is a sequence of positive real numbers satisfying certain conditions but this does not seem to give further information. Note that the terms $(e - 1)$ play a crucial role in all proofs of Cohen's factorization theorem because for each c there is an e with $\|(e - 1)c\|$ small. For commutative algebras the standard proof can be adapted to give most of Theorem 1. When $d = 1$, the b_n in this proof can be used in the proof of the main result in [6].

We shall often need to choose an element e in a Banach algebra such that $\|ec_j - c_j\|$ and $\|ez_j - z_j\|$ are small for a finite number of j, where c_j is in A and z_j is in X_0, by using a lemma that enables us to choose e for one element in a left Banach A module. We make the choice by applying the lemma to the left Banach A module $A \oplus \cdots \oplus A \oplus X \oplus \cdots \oplus X$, and the element $(c_1, \ldots, c_n, x_1, \ldots, x_n)$. In the proofs we shall assume that our left and right Banach A modules satisfy $\| cz \| \leqslant \| c \| \cdot \| z \|$ and $\| wc \| \leqslant \| c \| \cdot \| w \|$ for c in A, and z and w in the modules.

3. Proof of Theorem 1

In the three lemmas A is a Banach algebra, X is a left Banach A module, X_0 is the closed linear span of $A \cdot X = \{cz : c \in A,\ z \in X\}$, A_1 is the Banach algebra obtained by adjoining an identity to A, and $d \geqslant 1$. We shall also apply the lemmas to right Banach A modules. The right form of the lemma may be proved in a similar manner or obtained from the left form by considering the reversed product on the algebra and module [3, p. 6]. The first lemma is the standard starting point in the proofs of the stronger forms of Cohen's factorization theorem, and we omit the proof as it is proved in [10; 2, Lemma 1]. It ensures that a bounded approximate identity for A is a bounded approximate identity for X_0.

Lemma 3. *If $z_1, \ldots, z_n$ are in X_0 and $\epsilon > 0$, then there are $c_1, \ldots, c_m$ in A and $\eta > 0$ such that $\| ez_j - z_j \| < \epsilon$ for $j = 1, \ldots, n$ for each e in A with $\| e \| \leqslant d$ and $\| ec_k - c_k \| < \eta$ for $k = 1, \ldots, m$.*

Lemma 4. *Let n be a positive integer and let $\epsilon > 0$.*

(a) *If f is in A, then there is an $\eta > 0$ such that $\| f^k + (e-1)^k - (f + (e-1))^k \| < \epsilon$ for $k = 1, \ldots, n$, and for all e in A with $\| e \| \leqslant d$ and $\| f(e-1) \| + \|(e-1)f\| < \eta$.*

(b) *If c is in A_1 and x is in X_0, then there is an $\eta > 0$ such that $\| c^k x - (c + (e-1))^k x \| < \epsilon$ for $k = 1, \ldots, n$ and for all e in A with $\| e \| \leqslant d$ and $\|(e-1)x\| + \|(e-1)f\| < \eta$ where $c = f + \mu 1$ with $f \in A$ and $\mu \in C$.*

Proof. (a) Multiplying out the power $(f + (e-1))^k$ and using the norm inequalities, we obtain

$$\| f^k + (e-1)^k - (f + (e-1))^k \| \leqslant \sum_{j=1}^{k-1} \| f \|^{k-1-j} \cdot \| e - 1 \|^{j-1} \left(\binom{k}{j} - 1 \right) \{ \|(e-1)f\| + \| f(e-1) \| \}$$

for $k = 1, \ldots, n$. We now choose $\eta > 0$ such that

$$\sum_{j=1}^{k-1} \binom{k}{j} \| f \|^{k-1-j} (d+1)^{j-1} < \eta^{-1} \epsilon$$

for $k = 1, \ldots, n$.

(b) Multiplying out the powers $(\mu 1 + f)^k$ and $(\mu 1 + (f + e - 1))^k$ and using the norm inequalities, we obtain

$$\| c^k x - (c + (e-1))^k x \| \leqslant \sum_{j=1}^{k} \binom{k}{j} | \mu |^{k-j} \| f^j \cdot x - (f + (e-1))^j \cdot x \|$$

for $k = 1, \ldots, n$. Note that we could have applied the binomial theorem to obtain this because μ is in the center of A_1, but that we could not apply the binomial theorem in (a) as e and f may not commute. Now there is a small positive ν such that $\| f^j \cdot x - (f + (e-1))^j x \| < \nu$ for $j = 1, \ldots, n$ implies that $\| c^k \cdot x - (c + (e-1))^k \cdot x \| < \epsilon$ for $k = 1, \ldots, n$. For example, we could take $\nu = (1 + |\mu|)^{-n} \cdot \epsilon$.

Using a computation similar to that in (a) of expanding $(f + (e-1))^j$ and applying the norm inequalities we find that $\|(f + (e-1))^j x\|$ is less than or equal to a finite sum of numbers of the form

$$\| f \|^i \| e - 1 \|^{j-i-m-1} \|(e-1) f^m \cdot x \|,$$

where $i, m = 0, 1, \ldots, j-1$ with $i + m \leqslant j - 1$. The estimate $\| e - 1 \| \leqslant 1 + d$ enables us to choose $\eta > 0$ such that for $j = 1, \ldots, n$ we have $\| f^j \cdot x - (f + (e-1))^j \cdot x \| < \nu$. This completes the proof of the lemma.

We could have used (a) in the proof of (b) above by replacing $\|(e-1)x\| + \|(e-1)f\| < \eta$ by

$$\|(e-1)x\| + \|(e-1)f\| + \|f(e-1)\| < \eta.$$

LEMMA 5. *Let K be a bounded subset of the complex plane, and let $\epsilon > 0$.*

(a) *If $c = f + \mu 1 \in A_1$, then there is an $\eta > 0$ such that*

$$\| \exp t(c + (e-1)) - \exp tc \| \leqslant (\epsilon + \exp |t| (d+1) - 1) \cdot \exp \operatorname{Re}(t\mu).$$

for all $t \in K$ and all $e \in A$ with $\| e \| \leqslant d$ and $\|(e-1)f\| + \|f(e-1)\| < \eta$.

(b) *If $c = f + \mu 1 \in A_1$ and $x \in X$, then there is an $\eta > 0$ such that*

$$\| \exp t(c + (e-1)) \cdot x - \exp tc \cdot x \| < \epsilon$$

for all $t \in K$ and all $e \in A$ with $\| e \| \leqslant d$ and $\|(e-1)x\| + \|(e-1)f\| < \eta$.

Proof. (a) There is an L such that $|t| \leqslant L$ for all $t \in K$. For each complex number t we have

$$\| \exp t(c + (e-1)) - \exp tc \| = \exp \operatorname{Re}(t\mu) \cdot \| \exp t(f + (e-1)) - \exp tf \|,$$

so we study the latter factor. If $t \in K$ and $e \in A$ with $\| e \| \leqslant d$, then

$$\begin{aligned} \| \exp t(f + (e-1)) - \exp tf \| &\leqslant \sum_{k=1}^{\infty} \frac{|t|^k}{k!} \|(f + (e-1))^k - f^k \| \\ &< \frac{\epsilon}{3} + \sum_{k=1}^{n} \frac{|t|^k}{k!} \|(f + (e-1))^k - f^k \| \end{aligned}$$

provided $\sum_{k=n+1}^{\infty} (L^k/k!)(\| f \| + d + 1)^k < \epsilon/6$. By Lemma 4(a) we may choose $\eta > 0$ such that $\sum_{k=1}^{n} (L^k/k!) \|(f + (e - 1))^k - f^k - (e - 1)^k \| < \epsilon/3$ provided $\| e \| \leqslant d$ and $\| f(e - 1)\| + \|(e - 1)f \| < \eta$. If η and e are chosen like this, then

$$\| \exp t(f + (e - 1)) - \exp tf \| < \frac{2\epsilon}{3} + \sum_{k=1}^{n} \frac{| t |^k}{k!} (d + 1)^k$$
$$< \epsilon + \exp | t | \cdot (d + 1) - 1.$$

(b) The proof of this is similar to (a), and slightly simpler as the term $\exp | t | (d + 1)$ and factor $\exp \mathrm{Re}(\mu t)$ do not occur. It is a straightforward application of Lemma 4(b), and we omit this proof.

Remarks 6. The exponential function in the above lemma may be replaced by any entire function provided that in (a) $\mu = 0$.

If $\beta > 0$, let $\Delta(\beta) = \{t \in \mathbb{C} : | t | \leqslant \beta\}$ and let $D(\beta) = \{t \in \mathbb{C} : | t | \leqslant \beta$ and $\mathrm{Re} t \geqslant \beta^{-1}\}$. After an initial normalization and definition of a sequence required in the proof, the proof of Theorem 1 falls into two parts. We inductively choose a sequence (b_n) in A_1 to satisfy certain conditions, and in the second part of the proof we use this sequence to define a^t, x_t, and y_t and check that the conclusions are satisfied.

Proof of Theorem 7. We shall assume that $\| x \| \leqslant 1$ and $\| y \| \leqslant 1$ and that $\delta < 1$. We choose an increasing sequence of positive real numbers β_n that tend to infinity such that

$$\Delta(\beta_1) \supseteq C, \qquad \beta_1 > 1, \qquad \text{and that} \qquad 1 + \exp m(d + 1) \leqslant \alpha_t$$
$$\text{for all} \quad t \geqslant \beta_m \text{ and each positive integer } m. \tag{1}$$

The choice of the sequence (β_n) with these properties is possible because $\alpha_t \to \infty$ as $t \to \infty$. A similar sequence is chosen in the proof of Theorem 1 in [2].

By induction we shall choose sequences $e_1, e_2, \ldots$ in A, and $b_0, b_1, \ldots$ in $A_1 = A + \mathbb{C}1$ such that, for all positive integers n,

$$\| e_n \| \leqslant d, \tag{2}$$

$$b_0 = 1 \qquad \text{and} \qquad b_n = \exp\left(\sum_{j=1}^{n} (e_j - 1)\right), \tag{3}$$

$$\| b_{n-1}^{-t} \cdot x - b_n^{-t} \cdot x \| \leqslant \delta \cdot 2^{-n} \quad \text{and} \quad \| y \cdot b_{n-1}^{-t} - y \cdot b_n^{-t} \| \leqslant \delta \cdot 2^{-n} \tag{4}$$

for all $t \in \Delta(\beta_n)$, and

$$\| b_{n-1}^{t} - b_n^{t} \| \leqslant \delta \cdot 2^{-n} + \{\exp | t | (d + 1) - 1\} \cdot \exp - (n - 1) \mathrm{Re} t \tag{5}$$

for all $t \in \Delta(\beta_n)$.

We choose, e_1 to satisfy the conclusions of Lemma 5(a) and (b) simultaneously in the left- and right-module versions applied to the left and right Banach modules X and Y with $c = f = 0$ and $\epsilon = \delta/2$. This choice is possible using Lemma 3 because A has a bounded approximate identity bounded by d. Then e_1 and b_1 satisfy (2)–(5). Suppose $e_1, \ldots, e_n$ and $b_1, \ldots, b_n$ have been chosen to satisfy (2)–(5). We choose e_{n+1} to satisfy the conclusions of Lemma 5(a) and (b) applied to the left and right Banach A modules X and Y with $c = -n + f$, $f = \sum_{j=1}^{n} e_j$, $\epsilon = \delta \cdot 2^{-n-1} \cdot \exp(-n\beta_{n+1})$, and $K = \Delta(\beta_{n+1})$. Because A has a bounded approximate identity bounded by d, we may choose e_{n+1} in A such that $\| e_{n+1} \| \leqslant d$ and

$$\|(e_{n+1} - 1)x\| + \| y(e_{n+1} - 1)\| + \|(e_{n+1} - 1)f\| + \|f(e_{n+1} - 1)\| < \eta,$$

where η is the smallest of the four values given by Lemma 5 with our initial conditions.

We now use conditions (2)–(5) to construct a^t, x_t, and y_t and to check that these elements satisfy the conclusions. By (4) the sequence $(b_n^{-t} \cdot x)$ is Cauchy in X uniformly in t in $\Delta(\beta_m)$ for each positive integer m. Thus $\lim_{n\to\infty} b_n^{-t} \cdot x$ exists for all $t \in H$, and we denote this limit by x_t. Now the sequence $(b_n^{-t} \cdot x)$ converges to x_t uniformly in $t \in \Delta(\beta_m)$ for each positive integer m. Because $t \mapsto b_n^{-t} \cdot x$ is analytic in H, $t \mapsto x_t$ is analytic on $\operatorname{Int} D(\beta_m)$ for each m. Hence $t \mapsto x_t$ is analytic on H. The function $t \mapsto y_t$ is obtained in a similar way. From (5) we obtain

$$\| b_{n-1}^t - b_n{}^t \| \leqslant \exp\{\beta_m(d + 1) - (n - 1)\,\beta_m^{-1}\} + \delta \cdot 2^{-n}$$

for all $t \in D(\beta_m)$ and all $n \geqslant m$. Using this inequality as we used inequality (4) above we define $a^t = \lim_{n\to\infty} b_n{}^t$ for all $t \in H$, and obtain $t \mapsto a^t$ is analytic.

We now check the conclusions of Theorem 1. The property $x = a^t \cdot x_t$ follows from the equation $x = b_n{}^t \cdot (b_n^{-t} \cdot x)$ and the definitions of a^t and x_t. The equalities $a^{t+s} = a^t \cdot a^s$ and $a^t \cdot x_{s+t} = x_s$ both follow from the corresponding results with b_n in place of a and $b_n^{-t} \cdot x$ in place of x_t. If $d = 1$ and $t \in \mathbb{R}^+$, then

$$\| b_n{}^t \| = \left\| \exp t \sum_{j=1}^{n} (e_j - 1) \right\| \leqslant \exp - nt \cdot \exp nt = 1$$

for all positive integers n. In this case $\| a^t \| \leqslant 1$ for all $t > 0$ proving (iv). We now prove (v). Let $\epsilon > 0$. There is a positive integer m such that $2^{-m+2} \cdot \delta < \epsilon$. By (4) we have $\| b_{n-1}^t \cdot x - b_n{}^t \cdot x \| \leqslant \delta \cdot 2^{-n}$ for all $n \geqslant m$ and all $t \in \Delta(\beta_m)$ so that

$$\| a^t \cdot x - b_m{}^t \cdot x \| \leqslant 2^{-m+1} \cdot \delta < \epsilon/2$$

for all $t \in \Delta(1) \cap H \subseteq \Delta(\beta_m)$. Thus

$$\| a^t \cdot x - x \| \leqslant \epsilon/2 + \| b_m{}^t \cdot x - x \|$$
$$\leqslant \epsilon/2 + \sum_{j=1}^{\infty} \frac{|t|^j}{j!} m^j(d+1)^j$$

for all $t \in \Delta(1) \cap H$. Hence there is a $\nu > 0$ such that $\| a^t \cdot x - x \| < \epsilon$ for all $t \in H$ with $|t| < \nu$. This completes the proof of (v).

By (5) and the sum of a convergent geometric series, we have

$$\| a^t \| \leqslant 1 + \sum_{n=1}^{\infty} \| b_{n-1}^t - b_n{}^t \|$$
$$\leqslant 1 + \delta + \{\exp |t| (d+1) - 1\} \cdot \{1 - \exp - \mathrm{Re} t\}^{-1}$$

for $|t| \leqslant \beta_1$. The function of t on the right of this inequality has maximum $1 + \delta + (e^{d+1} - 1)(1 - e^{-1})^{-1}$ on the interval $(0, 1]$ because its derivative is positive on $(0, 1]$. Alternatively bound $\| a^t \|$ using continuity and that the limit of the right-hand side exists as $t \to 0$ with t in $(0, 1]$. This proves (vi).

We choose β_1 so that $C \subseteq \Delta(\beta_1)$. Thus

$$\| x - x_t \| \leqslant \sum_{n=1}^{\infty} \| b_{n-1}^{-t} \cdot x - b_n^{-t} \cdot x \| \leqslant \delta$$

for $t \in C$ by (4).

If $t \in H \cap \{\Delta(\beta_{m+1}) \backslash \Delta(\beta m)\}$, then

$$\| x_t \| \leqslant \| b_m^{-t} \cdot x \| + \sum_{j=m+1}^{\infty} \| b_{j-1}^{-t} \cdot x - b_j^{-t} \cdot x \|$$
$$\leqslant \| b_m^{-t} \cdot x \| + 2^{-m} \cdot \delta$$
$$\leqslant \exp |t| \cdot (d+1)\, m + 2^{-m} \cdot \delta$$

by (4). Since $|t| \geqslant \beta_m$, by (1) we have $1 + \exp m(d+1) \leqslant \alpha_t$ so that $\| x_t \| \leqslant \alpha_{|t|}{}^{|t|}$. This proves inequality (viii).

For each n and t, $b_n^{-t} \cdot x \in (A \cdot x)^-$ because $x \in (A \cdot x)^-$ by Lemma 1. Hence $x_t \in (A \cdot x)^-$ for all $t \in H$. This completes the proof of Theorem 1.

4. Applications of Theorem 1

Theorem 8. *Let A be a separable Banach algebra with bounded approximate identity. Then there is a commutative bounded approximate identity in A, and an equivalent Banach algebra norm on A for which this approximate identity is bounded by* 1.

Proof. Let (x_n) be a countable dense subset in A. Then there are elements x and y in A such that $x_n \in xA \cap Ay$ for all n [10, p. 269]. The elements x and y

may be obtained by applying Theorem 1 to the Banach A bimodule $X = \{(a_n): a_n \in A,\ a_n \to 0 \text{ as } n \to \infty\}$ in the usual way. By Theorem 1 we obtain the function $t \mapsto a^t$ for x and y in the Banach A bimodule A. By Theorem 1(v), (vi), $a^t \cdot x \to x$ and $y \cdot a^t \to y$ as $t \to 0$, and $\{\| a^t \| : 0 < t \leqslant 1\}$ is bounded. This and $(xA)^- = A = (Ay)^-$ implies that $\{a^t : 0 < t \leqslant 1\}$ is a commutative bounded approximate identity for A. Let $m = \log \| a^1 \|$. Then $\{a^t \cdot e^{-mt} : t \in \mathbb{R}^+\}$ is a bounded multiplicative semigroup in A, and as $t \to 0$ it is a bounded approximate identity for A since $e^{-mt} \to 1$ as $t \to 0$. A standard renorming of A gives an equivalent Banach algebra norm $|\cdot|$ on A with $| a^t | \leqslant 1$ for all $t \in \mathbb{R}^+$ [3, Theorem 1, p. 18]. This completes the proof.

Remarks 9. The hypothesis of separability in the above theorem may be replaced by the assumption that there exist elements x and y in A such that $(xA)^- = A = (Ay)^-$ (see [8]). If A is a Banach algebra of operators on some Banach space Z, then the renorming may be achieved by an equivalent renorming of Z and using the new operator norm on A [4, Lemma 3, p. 90]. The properties of the approximate identity in Theorem 8 are similar to those of the Poisson kernel listed in [14, p. 62].

THEOREM 10. *Let A be a separable Banach algebra. Then A has a bounded approximate identity bounded by* 1 *if and only if there is a homomorphism θ from $L^1(\mathbb{R}^+)$ into A such that $\theta(L^1(\mathbb{R}^+)) \cdot A = A = A \cdot \theta(L^1(\mathbb{R}^+))$ and $\| \theta \| = 1$.*

Proof. If A has a bounded approximate identity bounded by 1, then we choose x and y in A as in the proof of the previous theorem. A similar application of Theorem 1 gives a norm continuous one-parameter semigroup $t \mapsto a^t$ from $\mathbb{R}^+$ into A such that $x \in a^t \cdot A$ and $y \in A \cdot a^t$ for all t. Further $\| a^t \| \leqslant 1$ for all $t \in \mathbb{R}^+$ by Theorem 1(iv). We define $\theta: L^1(\mathbb{R}^+) \to A : f \mapsto \int_0^\infty f(t) a^t \, dt$. Straight forward calculations show that θ is a norm-reducing homomorphism from $L^1(\mathbb{R}^+)$ into A [11, Theorem 15.2.1, p. 436]. We may regard A as a left and right Banach $L^1(\mathbb{R}^+)$ module by defining $f \cdot a = \theta(f)a$ and $a \cdot f = a\theta(f)$ for all $a \in A$. By a standard corollary of the Cohen's factorization theorem [10, p. 268] that is contained in the statement of Theorem 1, $\theta(L^1(\mathbb{R}^+)) \cdot A$ and $A \cdot \theta(L^1(\mathbb{R}^+))$ are closed linear subspaces of A. To show that they are equal to A it is sufficient for us to prove that $x \in \{\theta(L^1(\mathbb{R}^+)) \cdot x\}^-$ because $(x \cdot A)^- = A$. Let f_n be n times the characteristic function of the interval $(0, 1/n)$ in $\mathbb{R}$. Then $\| x - \theta(f_n)x \| = \| \int_0^\infty f_n(t) \cdot (x - a^t x) \, dt \| \leqslant n \int_0^{1/n} \| x - a^t \cdot x \| \, dt$ for all positive integers n. By Theorem 1(v) we have $\| x - \theta(f_n)x \| \to 0$ as $n \to \infty$.

Conversely if θ exists, and if $b = \theta(g)c$ is in A, then $\lim_{n\to\infty} \theta(f_n)b = \lim_{n\to\infty} \theta(f_n * g)c = \theta(g)c = b$. Hence A has a bounded approximate identity bounded by 1, and the proof is complete.

Remarks 11. The sequence (f_n) used in the proof could be replaced by any bounded approximate identity bounded by 1 in $L^1(\mathbb{R}^+)$.

In the above theorem we have not excluded the possibility that A is the

Banach algebra of complex numbers $\mathbb{C}$, and so the kernel of θ could be a maximal ideal in $L^1(\mathbb{R}^+)$. The following argument of G. R. Allan and R. J. Loy shows that if θ is defined by $\theta(f) = \int_0^\infty f(t)\, a^t\, dt$ as in the above proof, then the kernel of θ cannot be a closed ideal of the form $M_\alpha = \{f \in L^1(\mathbb{R}^+) : f = 0 \text{ a.e. on } (0, \alpha]\}$ for a positive real number α. Suppose that $\ker \theta = M_\alpha$ for some $\alpha \in \mathbb{R}^+$. Let $s > \alpha$, and let g_n be the characteristic function of $[s - 1/n, s + 1/n]$. Since $t \to a^t$ is continuous, a standard calculation shows that

$$a^s = \lim_{n\to\infty} \int_0^\infty 2ng_n(t)\, a^t\, dt = 0.$$

This contradicts $x = a^s x_s \neq 0$.

Let ω: $[0, \infty) \to \mathbb{R}^+$ be a continuous function such that $\omega(s + t) \leqslant \omega(s)\, \omega(t)$ for all s, $t \geqslant 0$, and let $L^1(\mathbb{R}^+, \omega)$ denote the space of equivalence classes of Lebesgue measurable complex valued functions f such that $\int_0^\infty |f(t)|\, \omega(t)\, dt = \|f\|$ is finite. With product the convolution $(f * g)(s) = \int_0^s f(s - t)\, g(t)\, dt$, the Banach space $L^1(\mathbb{R}^+, \omega)$ becomes a Banach algebra [3, Example 21, p. 8]. If $t \mapsto a^t$ is chosen in a Banach algebra A with bounded approximate identity to satisfy Theorem 1, we let $\omega(t) = \| a^t \|$ for all $t > 0$. We may now define $\theta: L^1(\mathbb{R}^+, \omega) \to A$ by $\omega(f) = \int_0^\infty f(t) a^t\, dt$. As in Theorem 10, θ is a norm-reducing homomorphism from $L^1(\mathbb{R}^+, \omega)$ into A, and the working above shows that $\ker \theta$ is not an M_α. If A is a radical Banach algebra the spectral radius formula implies that $\omega(t)^{1/t} \to 0$ as $t \to \infty$, and so $L^1(R^+, \omega)$ is a radical Banach algebra. Is θ one-to-one in this case?

Dales [7] has proved that assuming the continuum hypothesis there is a discontinuous homomorphism from $C_0(\Omega)$, the algebra of continuous complex valued functions vanishing at infinity on an infinite noncompact locally compact Hausdorff space Ω, into $L^1(\mathbb{R}^+, \omega)$ if $\omega(t)^{1/t} \to 0$ as $t \to \infty$. Esterle [9] has proved that there is a discontinuous homomorphism from $C_0(\Omega)$ into a radical Banach algebra with a bounded approximate identity. Theorem 1 and the above remarks show that in this form Dales' theorem implies Esterle's theorem in this form.

Acknowledgments

I am grateful to B. E. Johnson for telling me of his factorization result, to J. Esterle for preprints, and to G. R. Allan, G. Dales and R. J. Loy for stimulating conversations and suggestions.

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